

A New Algorithm in Spectral Analysis and Band-Limited Extrapolation

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Abstract—If only a segment of a function $f(t)$ is given, then its Fourier spectrum $F(\omega)$ is estimated either as the transform of the product of $f(t)$ with a time-limited window $w(t)$, or by certain techniques based on various *a priori* assumptions. In the following, a new algorithm is proposed for computing the transform of a band-limited function. The algorithm is a simple iteration involving only the fast Fourier transform (FFT). The effect of noise and the error due to aliasing are determined and it is shown that they can be controlled by early termination of the iteration.

The proposed method can also be used to extrapolate band-limited functions.

I. INTRODUCTION

A CENTRAL PROBLEM in Fourier analysis and spectral estimation is the determination of the transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt \quad (1)$$

of a signal $f(t)$ in terms of a finite segment

$$g(t) = f(t)p_T(t), \quad p_T(t) = \begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases} \quad (2)$$

of $f(t)$. The known estimation methods can be separated into the following two classes.

1) (See Blackman-Tukey [1].) The available segment $g(t)$ is multiplied by a "window" $w(t)$ and the integral

$$\begin{aligned} F_w(\omega) &= \int_{-T}^T w(t)f(t)e^{-j\omega t} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - y)W(y) dy \end{aligned} \quad (3)$$

is used as the estimator of $F(\omega)$. The pair $w(t) \leftrightarrow W(\omega)$ is so chosen as to minimize in some sense the resulting error $F_w(\omega) - F(\omega)$ [2], [3]. In this approach, the inverse transform $f(t)w(t)$ of the estimator $F_w(\omega)$ is zero for $|t| > T$.

2) The unknown part of $f(t)$ is not assumed to be zero as in 1) but it is determined by some form of extrapolation. This is possible only if various *a priori* assumptions about form of $F(\omega)$ are made.

Class 2) includes the method of maximum entropy [4] (see Appendix A) and the techniques for extrapolating band-limited functions.

Band-Limited Functions

We shall say that a function $f(t)$ is band-limited, if its energy E is finite and its transform $F(\omega)$ is zero outside a finite interval

$$F(\omega) = 0, \quad \text{for } |\omega| > \sigma. \quad (4)$$

A band-limited function is analytic in the entire t -axis [5], hence its Taylor-series expansion yields $f(t)$ for every t . We have thus an extrapolation method (analytic continuation) based on (4), however, as it is well known, we cannot use it for recovering $f(t)$ from real data.

In this paper, we develop a new method for computing $f(t)$ and its transform $F(\omega)$ from the given segment $g(t)$. The method is based on a simple efficient algorithm involving only discrete Fourier series and a fast Fourier transform (FFT). The algorithm is presented in analog form. The numerical implementation is carried out digitally but, since the analog to digital conversion is well known, the details are omitted.

We note that the proposed algorithm (Section II) has been used by Gerchberg in a recent paper on picture processing [6].

Another method of extrapolating band-limited functions is based on the expansion of $g(t)$ into a series of prolate spheroidal functions [7]. This method is compared with ours in Section V.

II. AN ITERATION METHOD FOR COMPUTING THE FOURIER TRANSFORM OF A BAND-LIMITED FUNCTION

We wish to determine the Fourier transform $F(\omega)$ of a band-limited function $f(t)$ (Fig. 1) in terms of the finite segment $g(t)$ of $f(t)$ [see (2)]. The proposed method is an iteration starting with the transform

$$G(\omega) = G_0(\omega) = \int_{-T}^T g(t)e^{-j\omega t} dt$$

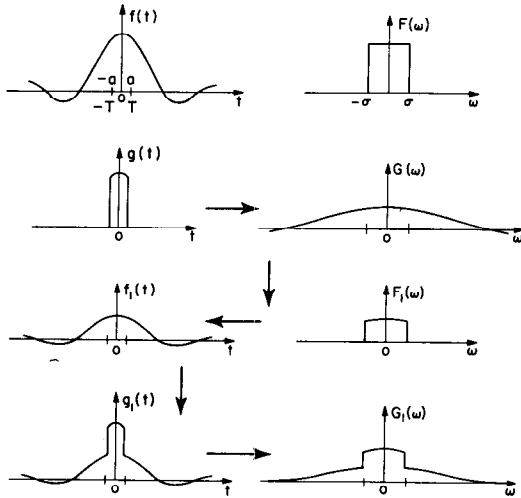
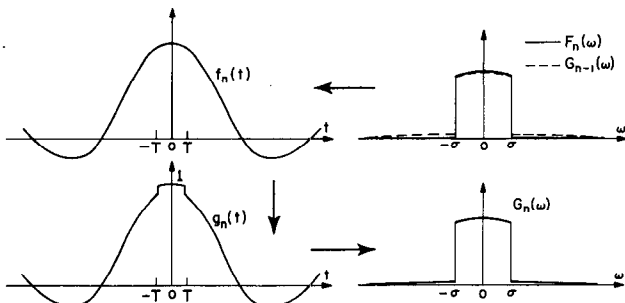
of the given segment $g(t) = g_0(t)$. The n th iteration step proceeds as follows.

We form the function

$$F_n(\omega) = G_{n-1}(\omega)p_\sigma(\omega), \quad p_\sigma(\omega) = \begin{cases} 1, & |\omega| < \sigma \\ 0, & |\omega| > \sigma \end{cases} \quad (5)$$

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Fig. 1. Unknown pair $f(t) \leftrightarrow F(\omega)$ and signals of first iteration.Fig. 2. Signals of n th iteration.

obtained by truncating $G_{n-1}(\omega)$ as in Fig. 2, and compute its inverse transform

$$f_n(t) = \int_{-\sigma}^{\sigma} F_n(\omega) e^{j\omega t} d\omega. \quad (6)$$

We, next, form the function

$$g_n(t) = f_n(t) + [f(t) - f_n(t)]p_T(t) = \begin{cases} g(t), & |t| < T \\ f_n(t), & |t| > T \end{cases} \quad (7)$$

obtained by replacing the segment of $f_n(t)$ in the interval $(-T, T)$ by the known segment $g(t)$ of $f(t)$. The n th step ends by computing the transform

$$G_n(\omega) = \int_{-\infty}^{\infty} g_n(t) e^{-j\omega t} dt \quad (8)$$

of the function $g_n(t)$ so formed.

We note that the function $f_n(t)$ is band-limited and it is given by

$$f_n(t) = g_{n-1}(t) * \frac{\sin \sigma t}{\pi t}. \quad (9)$$

This follows from (5) and the convolution theorem [8]. From (9) we conclude that $f_n(t)$ can be obtained as the output of an ideal low-pass filter with input $g_{n-1}(t)$. A two-input gate at the output of the filter, switching from $f_n(t)$ to

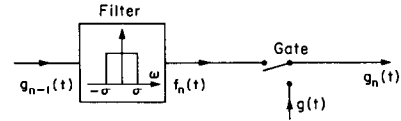
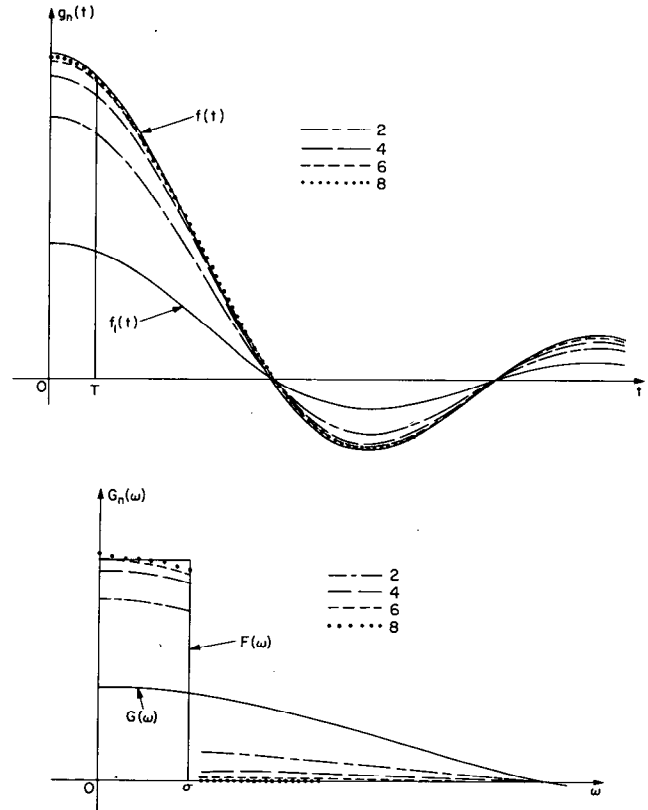


Fig. 3. Analog simulation of iteration.

Fig. 4. Computer output of $g_n(t)$ and $G_n(\omega)$ for $f(t) = \sin \sigma t / \pi t$ and $n = 2, 4, 6$, and 8 .

the given segment $g(t)$ of $f(t)$ as in (7), yields $g_n(t)$ (Fig. 3).

It will be shown in the next section that the functions $f_n(t)$ and $F_n(\omega)$ tend to $f(t)$ and $F(\omega)$, respectively, as $n \rightarrow \infty$. The iteration is thus a method for computing $F(\omega)$ and for extrapolating a band-limited function $f(t)$.

Numerical Illustration

We have applied the method to the pair

$$f(t) = \frac{\sin \sigma t}{\pi t} \leftrightarrow F(\omega) = p_\sigma(\omega)$$

choosing for T the value $\pi/5\sigma$. The numerical results are shown in Fig. 4 for $n = 2, 4, 6, 8$. As we see from the figure, although the data interval is only $1/5$ of the main lobe of $f(t)$, the convergence of $F_n(\omega)$ to the unknown spectrum $p_\sigma(\omega)$ is remarkably rapid. With this T , the method of windows would yield no satisfactory estimate of $F(\omega)$. We should point out, however, that the speed of convergence observed for this example is due to the fact that in the expansion (11) of the function $\sin \sigma t / \pi t$, only the first few terms are significant. As we show in Section IV, if this is not the case, the convergence is slow.

Mean-Square Error Reduction

From (5), (7), and Parseval formula [8] it follows that

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega) - G_{n-1}(\omega)|^2 d\omega \\ & > \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |F(\omega) - F_n(\omega)|^2 d\omega \\ & = \int_{-\infty}^{\infty} |f(t) - f_n(t)|^2 dt > \int_{-\infty}^{\infty} |f(t) - g_n(t)|^2 dt \\ & = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega) - G_n(\omega)|^2 d\omega. \end{aligned} \quad (10)$$

Thus the use of the available information about $f(t)$ (band-limited for $|\omega| > \sigma$ and known for $|t| < T$) results in a reduction of the mean-square error twice at each iteration step.

III. CONVERGENCE

To prove the convergence of the iteration, we expand the band-limited function $f(t)$ into a series

$$f(t) = \sum_{k=0}^{\infty} a_k \phi_k(t) \quad (11)$$

where $\phi_k(t)$ are the prolate spheroidal wave functions [see (68)].

Theorem 1

The function $f_n(t)$ of the n th iteration is given by

$$f_n(t) = f(t) - \sum_{k=0}^{\infty} a_k (1 - \lambda_k)^n \phi_k(t) \quad (12)$$

where λ_k are the eigenvalues of (63) corresponding to the eigenfunctions $\phi_k(t)$.

Proof: Suppose, first, that

$$f(t) = \phi_k(t).$$

We shall show by induction that

$$f_n(t) = A_n \phi_k(t), \quad \text{where } A_n = 1 - (1 - \lambda_k)^n. \quad (13)$$

Indeed, (13) is true for $n = 1$. Suppose that it is true for some $n \geq 1$. It, then, follows from (7), (9), and (63) that

$$g_n(t) = A_n \phi_k(t) + (1 - A_n) \phi_k(t) p_T(t). \quad (14)$$

Inserting into (9) and using (63) and (67), we obtain

$$f_{n+1}(t) = A_n \phi_k(t) + (1 - A_n) \lambda_k \phi_k(t) \quad (15)$$

hence,

$$f_{n+1}(t) = A_{n+1} \phi_k(t)$$

where

$$A_{n+1} = A_n + (1 - A_n) \lambda_k, \quad n \geq 1. \quad (16)$$

Solving the recursion equation (16) with the initial condition $A_1 = \lambda_k$, we obtain (13).

Applying (13) to each term in (11), we conclude that

$$f_n(t) = \sum_{k=0}^{\infty} a_k [1 - (1 - \lambda_k)^n] \phi_k(t) \quad (17)$$

and (12) results.

Theorem 2

For any t ,

$$f_n(t) \rightarrow f(t) \quad \text{as } n \rightarrow \infty. \quad (18)$$

Proof: As we see from (12), the error

$$e_n(t) = f(t) - f_n(t) \quad (19)$$

is given by

$$e_n(t) = \sum_{k=0}^{\infty} a_k (1 - \lambda_k)^n \phi_k(t) \quad (20)$$

and its mean-square value E_n by

$$E_n = \int_{-\infty}^{\infty} e_n^2(t) dt = \sum_{k=0}^{\infty} a_k^2 (1 - \lambda_k)^{2n} \quad (21)$$

[see (70)]. We maintain that

$$E_n \rightarrow 0 \quad \text{with } n \rightarrow \infty. \quad (22)$$

Indeed, the energy

$$E = \sum_{k=0}^{\infty} a_k^2 \quad (23)$$

of $f(t)$ is finite by assumption; hence, given $\varepsilon > 0$, we can find an integer N such that

$$\sum_{k=N}^{\infty} a_k^2 < \varepsilon.$$

The eigenvalues λ_k are less than 1 and tend to zero monotonically as $k \rightarrow \infty$. Therefore, $1 - \lambda_k < 1$ and

$$1 - \lambda_k \leq 1 - \lambda_N, \quad \text{for } k \leq N.$$

From the above we conclude that

$$\begin{aligned} E_n &= \sum_{k=0}^N a_k^2 (1 - \lambda_k)^{2n} + \sum_{k=N}^{\infty} a_k^2 (1 - \lambda_k)^{2n} \\ &< (1 - \lambda_N)^{2n} \sum_{k=0}^N a_k^2 + \sum_{k=N}^{\infty} a_k^2 < (1 - \lambda_N)^{2n} E + \varepsilon \end{aligned}$$

and (22) results because ε is arbitrary and

$$(1 - \lambda_N)^{2n} \rightarrow 0 \quad \text{with } n \rightarrow \infty.$$

To complete the proof of the theorem, we observe that

$$|e_n(t)| \leq \sqrt{\frac{E_n \sigma}{\pi}}. \quad (24)$$

This follows from the fact that the transform of $e_n(t)$ is zero for $|\omega| > \sigma$ [9]. Thus $e_n(t) \rightarrow 0$ with $n \rightarrow \infty$ and (18) results.

We note for later use that [see (7), (12), and (66)]

$$G(\omega) = B \sum_{k=0}^{\infty} a_k \sqrt{\lambda_k} \phi_k(b\omega), \quad B = \sqrt{\frac{2\pi T}{\sigma}}, \quad b = \frac{T}{\sigma} \quad (25)$$

$$F_n(\omega) = B p_\sigma(\omega) \sum_{k=0}^{\infty} \frac{a_k}{\sqrt{\lambda_k}} [1 - (1 - \lambda_k)^n] \phi_k(b\omega) \quad (26)$$

$$G_n(\omega) = F_n(\omega) + B \sum_{k=0}^{\infty} a_k \sqrt{\lambda_k} (1 - \lambda_k)^n \phi_k(b\omega). \quad (27)$$

Reasoning similarly, we conclude that the function $F_n(\omega)$ and $G_n(\omega)$ tend to $F(\omega)$ with $n \rightarrow \infty$.

IV. ERROR ANALYSIS

The proposed method is subject to the following errors: truncation of the iteration; aliasing due to high frequencies in $F(\omega)$; noise; computer roundoff; analog to digital conversion. These errors depend on the nature of $f(t)$ and on the product σT . In the following, we examine their effect on the estimate $F_n(\omega)$ of $F(\omega)$. The analog to digital conversion error involving the evaluation of the Fourier integrals by discrete Fourier series is well known and will not be discussed.

Truncation Error

If the iteration is terminated at the n th step and the function $F_n(\omega)$ is used as the estimate of $F(\omega)$, the resulting mean-square error is given by [see (21)]

$$E_n = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |F(\omega) - F_n(\omega)|^2 d\omega = \sum_{k=0}^{\infty} a_k^2 (1 - \lambda_k)^{2n} \quad (28)$$

where λ_k are the eigenvalues of (63). As we see from Fig. 5, the λ_k 's increase as the product σT increases and they tend to 1 with $\sigma T \rightarrow \infty$ [10]. Thus if σT is large, the required number of iterations is small.

The error E_n depends also on the coefficients a_n . If n is small, then the contribution of the high-order terms in (28) is small. However, as n increases, their effect becomes relatively more significant because the difference $1 - \lambda_k$ tends to 1 with $k \rightarrow \infty$.

From (27) it follows after some thought that the energy I_n of $G_n(\omega)$ outside the band $(-\sigma, \sigma)$ is given by

$$I_n = \frac{1}{2\pi} \int_{|\omega| > \sigma} |G_n(\omega)|^2 d\omega = \sum_{k=0}^{\infty} a_k^2 \lambda_k (1 - \lambda_k)^{2n+1}. \quad (29)$$

Hence [see (21)],

$$I_n \leq \frac{1}{4} \sum_{k=0}^{\infty} a_k^2 (1 - \lambda_k)^{2n} = E_n/4 \quad (30)$$

because $\lambda_k(1 - \lambda_k) < \frac{1}{4}$. If, at the n th-iteration step, the energy I_n computed from (29), is large, then, the energy E_n of the error is also large. However, if I_n is small, E_n is not necessarily small. The inequality

$$E_{n+1} < \frac{\lambda_n}{1 - \lambda_n} I_n + \sum_{k > n} a_k^2 \quad (31)$$

obtained readily from (21) and (29), gives an upper bound for E_{n+1} but this is useful only if some information about the speed of convergence of the series (11) is available.

Suppose, finally, that $F(\omega) = 0$ for $|\omega| > \omega_0$ but the precise value of ω_0 is not given. In this case, the following

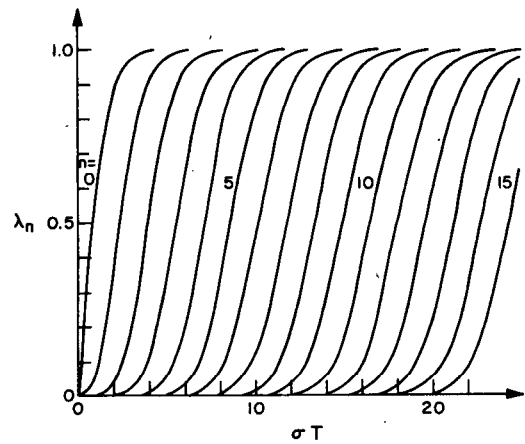


Fig. 5. First 15 eigenvalues λ_n of the integral equation

$$\int_{-\tau}^{\tau} \phi(\tau) \frac{\sin \sigma(t - \tau)}{\pi(t - \tau)} d\tau = \lambda \phi(t).$$

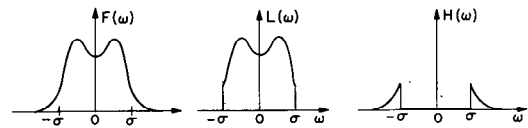


Fig. 6. Low- and high-frequency components of $F(\omega)$.

must be considered in the selection of the cutoff frequency σ . If σ is large, then $1 - \lambda_k$ is small; however, this does not lead to a reduction of the error E_n because an increase of σ beyond ω_0 results in an increase of the coefficients a_k . If σ is smaller than ω_0 , then as we show next, aliasing errors result. The optimum σ is a number close to ω_0 from above.

Aliasing Error

The convergence of $f_n(t)$ to the function $f(t)$ is based on the assumption that $F(\omega) = 0$ for $|\omega| > \sigma$. If $f(t)$ is not band-limited, then this is no longer true. However, as we shall show, the iteration properly terminated yields a useful estimate of $F(\omega)$ as long as the energy E_h of the high-frequency component $h(t)$ of $f(t)$ is sufficiently small.

We introduce the functions (Fig. 6)

$$\begin{aligned} L(\omega) &= \begin{cases} F(\omega), & |\omega| \leq \sigma \\ 0, & |\omega| > \sigma \end{cases} \\ H(\omega) &= \begin{cases} 0, & |\omega| \leq \sigma \\ F(\omega), & |\omega| > \sigma \end{cases} \end{aligned} \quad (32)$$

and their inverse transforms $l(t)$ and $h(t)$. Clearly,

$$\begin{aligned} F(\omega) &= L(\omega) + H(\omega) \\ f(t) &= l(t) + h(t). \end{aligned}$$

The iteration scheme is obviously linear in $f(t)$; therefore, $f_n(t)$ can be written as a sum

$$f_n(t) = l_n(t) + h_n(t) \quad (33)$$

where $l_n(t)$ and $h_n(t)$ are the contributions due to $l(t)$ and $h(t)$, respectively. As we see from (32), the function $l(t)$ is band-limited; hence, the component $l_n(t)$ of $f_n(t)$

tends to $l(t)$ with $n \rightarrow \infty$. It suffices, therefore, to examine the behavior of $h_n(t)$.

We expand the function $h(t)$ into a series in the interval $(-T, T)$

$$h(t) = \sum_{k=0}^{\infty} c_k \phi_k(t), \quad |t| \leq T \quad (34)$$

[see (69)]. Clearly,

$$\begin{aligned} \sum_{k=0}^{\infty} c_k^2 \lambda_k &= \int_{-T}^T h^2(t) dt < E_h \\ &= \frac{1}{2\pi} \int_{|\omega| > \sigma} |F(\omega)|^2 d\omega. \end{aligned} \quad (35)$$

From (17) it follows that

$$h_n(t) = \sum_{k=0}^{\infty} c_k (1 - \mu_k^n) \phi_k(t), \quad \mu_k = 1 - \lambda_k. \quad (36)$$

The function $h_n(t)$ is band-limited by construction and its energy equals

$$\begin{aligned} E_{nh} &= \sum_{k=0}^{\infty} c_k^2 (1 - \mu_k^n)^2 \\ &= \sum_{k=0}^{\infty} c_k^2 \lambda_k^2 (1 + \mu_k + \cdots + \mu_k^{n-1})^2. \end{aligned} \quad (37)$$

The numbers λ_k and μ_k are positive and less than 1; hence,

$$1 + \mu_k + \cdots + \mu_k^{n-1} < n$$

and

$$E_{nh} < n^2 \sum_{k=0}^{\infty} c_k^2 \lambda_k = n^2 E_h. \quad (38)$$

The above bound is too pessimistic; the actual value of E_{nh} might be considerably smaller.

Noise and Roundoff Error

Suppose that for $|t| < T$, the known signal is a sum

$$g(t) + y(t) \quad (39)$$

where $y(t)$ is a random process with autocorrelation

$$E\{y(t_1)y(t_2)\} = R(t_1, t_2).$$

At the n th iteration, $f_n(t)$ is replaced by $f_n(t) + x_n(t)$ where the random component $x_n(t)$ is due to $y(t)$. We shall assume that the noise $y(t)$ is white as in (78)

$$R(t_1, t_2) = Sp_T(t_1)\delta(t_1 - t_2). \quad (40)$$

We expand the transform

$$Y(\omega) = \int_{-T}^T y(t)e^{-j\omega t} dt \quad (41)$$

of $y(t)$ into a Karhunen-Loeve series in the interval $(-\sigma, \sigma)$ [see (74), (80), and (25)]

$$X_1(\omega) = Y(\omega)p_\sigma(\omega) = Bp_\sigma(\omega) \sum_{k=0}^{\infty} d_k \sqrt{\lambda_k} \phi_k(b\omega). \quad (42)$$

It is not difficult to see from (75) that

$$E\{d_k d_r^*\} = \begin{cases} S/\lambda_k, & k = r \\ 0, & k \neq r \end{cases} \quad (43)$$

$$E\{|X_1(\omega)|^2\} = 2S\pi b \sum_{k=0}^{\infty} \phi_k^2(b\omega) = 2ST. \quad (44)$$

The n th-iteration function $X_n(\omega)$ is given by the right side of (26) where the coefficients a_k are replaced by d_k . Hence, its mean-square value R_n is given by

$$\begin{aligned} R_n &= E\{|X_n(\omega)|^2\} \\ &= 2S\pi b \sum_{k=0}^{\infty} \left[\frac{1 - (1 - \lambda_k)^n}{\lambda_k} \right]^2 \phi_k^2(b\omega) \\ &= 2S\pi b \sum_{k=0}^{\infty} (1 + \mu_k + \cdots + \mu_k^{n-1})^2 \phi_k^2(b\omega). \end{aligned} \quad (45)$$

This error is similar to the aliasing error E_{nh} in (37) and it is bounded by

$$R_n \leq 2STn^2 \quad (46)$$

as in (38).

The preceding analysis can be used to determine the effect of the roundoff error on the estimate $F_n(\omega)$ of $F(\omega)$. In the evaluation of the Fourier transform $G(\omega)$ of $g(t)$ by a sum, roundoff errors are introduced. If these errors are independent with standard deviation σ (smallest available digit), then, their effect on $F_n(\omega)$ is approximately equivalent to the presence of a noise term $y(t)$ as in (39) with $S = \sigma^2 \delta$ where δ is the sampling interval. The resulting mean-square error at the n th iteration is given by R_n as in (45). Similarly, the roundoff error introduced at the k th-iteration step causes a mean-square error in the evaluation of $F_n(\omega)$ equal to R_{n-k} .

V. COMPARISON WITH THE PROLATE SPHEROIDAL EXPANSION METHOD

We shall discuss briefly the method of prolate expansion for determining $F(\omega)$ from $g(t)$ and will compare it with our approach.

We expand the time-limited function $g(t)$ into a series as in (69). The coefficients a_k are given by

$$a_k = \frac{1}{\lambda_k} \int_{-T}^T g(t) \phi_k(t) dt. \quad (47)$$

We insert the numbers so obtained in the expansion (11). The resulting sum $f(t)$ equals the band-limited extrapolation $f(t)$ of the piece $g(t)$. The transform $F(\omega)$ of $f(t)$ is obtained from (25)

$$F(\omega) = B \sum_{k=0}^{\infty} \frac{a_k}{\sqrt{\lambda_k}} \phi_k(b\omega), \quad |\omega| \leq \sigma. \quad (48)$$

Thus to evaluate $F(\omega)$ with this method, we must store the sample values of the functions $\phi_k(t)$ and use them to evaluate the integral in (47). Our method involves only discrete Fourier series and makes use of available algorithms and the $N \log N$ economy of the FFT.

In most applications, the product σT is large. This assumption does not reduce the computations in the prolate method. In our method it leads to a rapid convergence that can be sensed by computing E_{nh} .

If noise or aliasing errors are present, no method permits the precise evaluation of $F(\omega)$. An approximate result is possible with the prolate method if the series (48) is truncated. In our approach, the iteration must be carried out to a value n such that the errors E_{nh} and R_n in (37) or (45) are sufficiently small. If n is too small, the truncation error E_n [see (28)] might be unacceptable. In a specific example, a compromise must be sought; the bounds (30), (38), and (46) give some guidance.

VI. RELATIONSHIP TO FREQUENCY-DEPENDENT WINDOWS AND INTEGRAL EQUATIONS

In the proposed iteration, the function $F_n(\omega)$ of the n th step is a linear functional of the segment $g(t)$. We denote by $y(\omega, \tau)$ the limit of $F_n(\omega)$ corresponding to $g(t) = \delta(t - \tau)$. Since

$$g(t) = \int_{-T}^T g(\tau) \delta(t - \tau) d\tau$$

it follows from the linearity of the iteration that

$$F(\omega) = \int_{-T}^T g(\tau) y(\omega, \tau) d\tau. \quad (49)$$

With

$$w(\omega, t) = y(\omega, t) e^{j\omega t}$$

(49) yields

$$F(\omega) = \int_{-T}^T w(\omega, t) g(t) e^{-j\omega t} dt. \quad (50)$$

This is of the form (3) with the difference that the window $w(\omega, t)$ is frequency dependent. Unlike the estimate $F_w(\omega)$ in (3), the inverse of $F(\omega)$ in (50) is not zero for $|t| > T$ but it equals the unknown segment of $f(t)$.

We shall next show that the function $y(\omega, \tau)$ in (49) is the Fourier transform of the resolving kernel of an integral equation. Since $f(t) = g(t)$ for $|t| < T$, the Fourier transform $F(\omega)$ of $f(t)$ can be written as a sum

$$F(\omega) = \int_{|t| < T} g(t) e^{-j\omega t} dt + \int_{|t| > T} f(t) e^{-j\omega t} dt. \quad (51)$$

But $F(\omega) = 0$ for $|\omega| > \sigma$ by assumption; hence,

$$\int_{|t| > T} f(t) e^{-j\omega t} dt = - \int_{-T}^T g(t) e^{-j\omega t} dt, \quad \text{for } |\omega| > \sigma. \quad (52)$$

This is a linear integral equation for the unknown segment of $f(t)$. Denoting by $k(t, \tau)$ its resolving kernel, we obtain

$$f(t) = \int_{-T}^T g(\tau) k(t, \tau) d\tau. \quad (53)$$

Taking Fourier transforms of both sides of (53) we obtain (49) where

$$y(\omega, \tau) = \int_{-\infty}^{\infty} k(t, \tau) e^{-j\omega t} dt. \quad (54)$$

The discrete form of (52) is a system of linear equations.

APPENDIX A

The Method of Maximum Entropy

In the numerical evaluation of $F(\omega)$, the integral in (1) is replaced by a sum

$$F(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-jna\omega}, \quad f_n = af(na). \quad (55)$$

This is the exact if $F(\omega) = 0$ for $|\omega| > \pi/a$. Otherwise, it is approximate subject to aliasing errors. Since only the segment $g(t)$ of $f(t)$ is known, the above sum must be estimated in terms of its $2M + 1$ coefficients

$$f_n = g_n = ag(na), \quad |n| \leq M = [T/a]. \quad (56)$$

The method of maximum entropy is developed for the estimation of power spectra [11], [12]. In this case, $F(\omega)$ is real and positive. The function $F(\omega)$ is so selected as to maximize the integral (entropy)

$$\int_{-\pi/a}^{\pi/a} \log F(\omega) d\omega$$

subject to the $2M + 1$ constraints (56). It is easy to show that [13], this assumption leads to the conclusion that $F(\omega)$ must be rational with no finite zeros. Since $F(\omega) \geq 0$, it follows from the Fejer-Riesz theorem [5] that

$$F(\omega) = \frac{p}{\left| 1 + \sum_{k=1}^M y_k e^{-jka\omega} \right|^2} \quad (57)$$

where all the zeros z_i of the polynomial

$$Y(z) = 1 + \sum_{k=1}^M y_k z^{-k}$$

are inside the unit circle.

Our problem is thus reduced to the determination of the $M + 1$ constants $p, y_1, \dots, y_M$. The number of independent equations in (56) is also $M + 1$ because $f_{-n} = f_n^*$.

We maintain that

$$\sum_{k=1}^M g_{n-k} y_k = -g_n, \quad n = 1, \dots, M \quad (58)$$

and

$$p = g_0 + \sum_{k=1}^M g_{-k} y_k. \quad (59)$$

Thus to find the y_k 's we solve the system (58). The special form g_{n-k} of the coefficient matrix (Toeplitz) simplifies the computations. Inserting the solution into (59), we obtain the constant p .

Proof: With

$$H(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} \quad (60)$$

we have [see (55)] $F(\omega) = H(e^{j\omega a})$ and (57) yields

$$H(z) \left(1 + \sum_{k=1}^M y_k z^{-k} \right) = \frac{p}{1 + \sum_{k=1}^M y_k^* z^k}. \quad (61)$$

The zeros $1/z_i^*$ of the denominator $Y^*(1/z^*)$ of (61) are outside the unit circle, hence, the inverse z -transform of the right side of (61) is zero for $n > 0$ and it equals p for $n = 0$. Equating the inverse z -transforms of both sides, we obtain

$$f_n + \sum_{k=1}^M f_{n-k} y_k = \begin{cases} p, & n = 0 \\ 0, & n > 0 \end{cases} \quad (62)$$

The above yields (58) for $n = 1, \dots, M$ [see (56)] and (59) for $n = 0$.

We note that the maximum entropy method can be considered as a method of estimating $F(\omega)$ in terms of g_n with the *a priori* assumption (Class 2)) that $F(\omega)$ is of the form (57).

APPENDIX B

The Prolate Spheroidal Functions

The eigenfunctions $\phi_k(t)$ of the equation

$$\int_{-T}^T \phi(\tau) \frac{\sin \sigma(t - \tau)}{\pi(t - \tau)} d\tau = \lambda \phi(t) \quad (63)$$

known as prolate spheroidal wave functions [7], are orthogonal in the intervals $(-\infty, \infty)$ and $(-T, T)$

$$\begin{aligned} \int_{-\infty}^{\infty} \phi_i(t) \phi_k(t) dt &= \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \\ \int_{-T}^T \phi_i(t) \phi_k(t) dt &= \begin{cases} \lambda_k, & i = k \\ 0, & i \neq k \end{cases} \end{aligned} \quad (64)$$

The eigenvalues λ_k are such that

$$1 > \lambda_0 > \dots > \lambda_k > \dots > 0 \quad \text{and} \quad \lambda_k \rightarrow 0, \quad \text{with } k \rightarrow \infty. \quad (65)$$

The functions $\phi_k(t)$ equal their own Fourier transforms properly truncated and scaled:

$$\begin{aligned} \phi_k(t) &\leftrightarrow \frac{B}{\sqrt{\lambda_k}} \phi_k(b\omega) p_\sigma(\omega), \\ B &= \sqrt{2\pi T/\sigma}, \quad b = T/\sigma \\ \phi_k(t) p_T(t) &\leftrightarrow B \sqrt{\lambda_k} \phi_k(b\omega). \end{aligned} \quad (66)$$

Thus the solutions of (63) are band-limited hence,

$$\int_{-\infty}^{\infty} \phi_k(\tau) \frac{\sin \sigma(t - \tau)}{\pi(t - \tau)} d\tau = \phi_k(t). \quad (67)$$

A band-limited function $f(t)$ can be expanded into a series in the interval $(-\infty, \infty)$

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} a_k \phi_k(t) \\ a_k &= \int_{-\infty}^{\infty} f(t) \phi_k(t) dt. \end{aligned} \quad (68)$$

An arbitrary function $g(t)$ can be expanded into a series in the interval $(-T, T)$

$$\begin{aligned} g(t) &= \sum_{k=0}^{\infty} a_k \phi_k(t) \\ a_k &= \frac{1}{\lambda_k} \int_{-T}^T g(t) \phi_k(t) dt. \end{aligned} \quad (69)$$

From the above and (64) we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} f^2(t) dt &= \sum_{k=0}^{\infty} a_k^2 \\ \int_{-T}^T g^2(t) dt &= \sum_{k=0}^{\infty} a_k^2 \lambda_k. \end{aligned} \quad (70)$$

We, finally, note that

$$\sum_{k=0}^{\infty} \phi_k^2(t) dt = \sigma/\pi. \quad (71)$$

This follows by expanding the band-limited function $\sin \sigma(t - x)/\pi(t - x)$ into a series as in (68) and using (63).

APPENDIX C

The Karhunen-Loève Expansion

Consider a stochastic process $y(t)$ with autocorrelation

$$R(t_1, t_2) = E\{y(t_1)y^*(t_2)\}.$$

With

$$\Gamma(u, v) = E\{Y(u)Y^*(v)\}$$

the autocorrelation of the Fourier transform $Y(\omega)$ of $y(t)$, it is easy to show that [14]

$$\Gamma(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t_1, t_2) \exp(-j(ut_1 - vt_2)) dt_1 dt_2. \quad (72)$$

With $\Gamma(u, v)$ as above, we form the equation

$$\int_{-\sigma}^{\sigma} \Gamma(u, v) \psi(v) dv = \gamma \psi(u) \quad (73)$$

and denote by $\psi_k(u)$ and γ_k its eigenfunctions and eigenvalues. It can be shown that in the expansion

$$Y(\omega) = \sum_{k=0}^{\infty} z_k \psi_k(\omega), \quad |\omega| \leq \sigma \quad (74)$$

the random variables z_k are orthogonal [14] and

$$E\{z_i z_k^*\} = \begin{cases} \gamma_k, & i = k \\ 0, & i \neq k \end{cases} \quad (75)$$

Nonstationary white noise: We say that the process $y(t)$ is white noise if

$$R(t_1, t_2) = q(t_1)\delta(t_1 - t_2) \quad (76)$$

where $q(t) \geq 0$ is its "average intensity." With $Q(\omega)$ the Fourier transform of $q(t)$, it follows from (72) that

$$\Gamma(u, v) = Q(u - v). \quad (77)$$

If $y(t) = n(t)p_T(t)$ and $n(t)$ is stationary white noise with autocorrelation $S\delta(t_1 - t_2)$, then

$$R(t_1, t_2) = Sp_T(t_1)\delta(t_1 - t_2). \quad (78)$$

In this case,

$$q(t) = Sp_T(t) \quad Q(\omega) = 2S \sin T\omega/\omega. \quad (79)$$

From the above and (77) it follows that the integral equation (73) takes the form

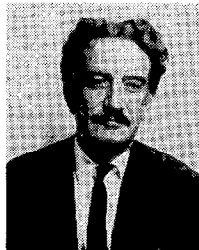
$$2S \int_{-\sigma}^{\sigma} \frac{\sin T(u - v)}{u - v} \psi(v) dv = \gamma \psi(u). \quad (80)$$

As we see from (63), its eigenfunctions $\psi_k(u)$ equal the prolate spheroidal functions $\phi_k(t)$ properly scaled.

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